Uitwerkingen Tentamen Complexe Analyse April 2012

1. $f(z) = x^3 - 3y^2 + 2x + i(3x^2y - y^3 + 2y)$

a) $u(x,y) = x^3 - 3y^2 + 2x$ and $v(x,y) = 3x^2y - y^3 + 2y$

f check in which points the CR-equations hold:

$$\frac{\partial u}{\partial x} = 3x^2 + 2 \quad ; \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 2$$

$$\frac{\partial u}{\partial y} = -6y \quad ; \quad \frac{\partial v}{\partial x} = 6xy .$$

So: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \iff y = 0$

$\boxed{8}$ So the Cauchy-Riemann equations hold on the set $\{(x,y) \in \mathbb{C} \mid y = 0\}$ (the real axis).

In every point in that set we have in addition

1) the first partial derivatives of u and v exist in a neighbourhood of that point

2) these partial derivatives are continuous at the point

Conclusion: $f(z)$ differentiable at every point in the set $y=0$.

b) $f(z)$ is analytic at z_0 if there exists an open neighbourhood of z_0 on which $f(z)$ is differentiable. This requires that the CR-equations hold on such open neighbourhood. However for no point

$\boxed{8}$ z_0 in $\mathbb{C}$ such open neighbourhood exists: given any $z_0 \in \mathbb{C}$ and any open neighbourhood of z_0, it always contains points where the CR equations do not hold.

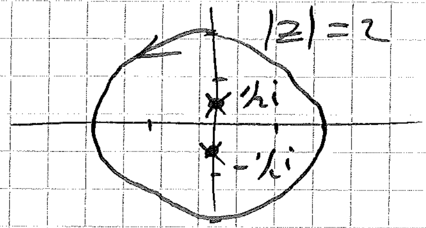
2. $$g(z) = \frac{e^{\pi z}}{4z^2+1}.$$

(a) $g(z)$ has singularities in those points z where $4z^2+1=0$.

Now $4z^2+1=0 \iff z = \pm\frac{1}{2}i$

By the residue theorem, $\int_{\Gamma_1} g(z)\,dz$ equals $2\pi i$ times the sum of the residues inside $|z|=2$.

We compute the residues



1) $z = \frac{1}{2}i$

$$\lim_{z\to\frac{1}{2}i} \left(z-\tfrac{1}{2}i\right) g(z) =$$

$\boxed{12}$ $$\lim_{z\to\frac{1}{2}i} \frac{e^{\pi z}}{4\left(z+\frac{1}{2}i\right)} = \frac{e^{\frac{1}{2}\pi i}}{4i} = \frac{i}{4i} = \frac{1}{4}$$

2) $z = -\frac{1}{2}i$

$$\lim_{z\to -\frac{1}{2}i} \left(z+\tfrac{1}{2}i\right) g(z) = \lim_{z\to -\frac{1}{2}i} \frac{e^{\pi z}}{4\left(z-\frac{1}{2}i\right)} = \frac{e^{-\frac{1}{2}\pi i}}{-4i}$$
$$= \frac{-i}{-4i} = 1/4$$

Conclusion: $$\int_{\Gamma_1} g(z)\,dz = 2\pi i \cdot \left(\frac{1}{4}+\frac{1}{4}\right) = \pi i$$

b) $\int_{\Gamma_2} g(z)\,dz = 0$, since $g(z)$ has <u>no</u>

$\boxed{6}$ <u>singularities</u> inside $|z| = \frac{1}{4}$

3) a) $f(z)$ is analytic, so has a power series
$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{around } z=0.$$

Now $$\frac{1}{z} - f(z) = \frac{1}{z} - a_0 - a_1 z - a_2 z^2 - \dots ,$$

$\boxed{10}$ which is a Laurent series. We see that $\frac{1}{z} - f(z)$ has <u>a pole at $z=0$</u> <u>with residue 1</u>

By the residue theorem $$\int_C \left(\frac{1}{z} - f(z)\right) dz = 2\pi i \cdot 1 = 2\pi i$$

b) $$\left| \int_C \left(\frac{1}{z} - f(z) \right) dz \right| \leq \max_{z \in C} \left| \frac{1}{z} - f(z) \right| \cdot L(C),$$

where $L(C)$ is the arclength of the circle $|z| = 1$.
This inequality yields:

$\boxed{6}$ $$\max_{z \in C} \left| \frac{1}{z} - f(z) \right| \geq \frac{1}{L(C)} \left| \int_C \left(\frac{1}{z} - f(z) \right) dz \right|$$

$$= |2\pi i| \cdot \frac{1}{2\pi} = 1$$

(Note that $L(C) = 2\pi$).

4) $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

(a) z_0 is a zero of $\tan z \iff \sin z_0 = 0$
$\iff z_0 = k\pi$ $(k \in \mathbb{Z})$. Since $f'(z) = \dfrac{1}{\cos^2 z}$

$\boxed{4}$ and $f'(z_0) = \dfrac{1}{\cos^2 k\pi} = 1 \neq 0$, these zeros have order 1.

(b) Singularities are the zeros of $\cos z$. Now

$\boxed{4}$ $\cos z = 0 \iff z = \frac{1}{2}\pi + k\pi$ $(k \in \mathbb{Z})$

(c) $z = \frac{1}{2}\pi$ is a pole since $\lim_{z \to \frac{1}{2}\pi} |\tan z| = \infty$

$\boxed{5}$

(d) Residue at $z = \frac{1}{2}\pi$ is equal to

$$\lim_{z \to \frac{1}{2}\pi} \left(z - \tfrac{1}{2}\pi \right) \frac{\sin z}{\cos z}$$

$\boxed{5}$

Now $\cos z$ has a Taylor series around $z = \frac{1}{2}\pi$:

$$\cos z = -\left(z - \tfrac{1}{2}\pi\right) + \frac{1}{3!}\left(z - \tfrac{1}{2}\pi\right)^3 - \frac{1}{5!}\left(z - \tfrac{1}{2}\pi\right)^5 + \ldots$$

$$\Rightarrow \lim_{z \to \frac{1}{2}\pi} \left(z - \tfrac{1}{2}\pi\right) \frac{\sin z}{\cos z} =$$

$$\lim_{z\to\frac{1}{2}\pi} \frac{\sin z}{-1+\frac{1}{3!}(z-\frac{1}{2}\pi)^2-\frac{1}{5!}(z-\frac{1}{2}\pi)^4+\dots} = \frac{\sin\frac{\pi}{2}}{-1}$$

$$= -1.$$

5) a) Stelling van Rouché: Let C be a simple closed contour and $f(z)$ and $\overset{h}{\cancel{g}}(z)$ analytic on and inside C. Assume the strict inequality

$\boxed{6}$ $$|h(z)| < |f(z)|$$

holds for each point z on the contour C

Then $f(z)$ and $f(z)+h(z)$ have the same number of zeros (counting multiplicities) inside C

b) Equation: $2z^5+8z-1=0$.

Define: $f(z) = 2z^5$; $h(z) = 8z-1$.

For z on the contour $|z|=2$ we have:

$$|h(z)| = |8z-1| \leq 8|z|+1 = 17$$

$\boxed{8}$ $$|f(z)| = 2|z|^5 = 64$$

Clearly $|h(z)| < |f(z)|$ for $|z|=2$.

Conclusion: $2z^5+8z-1=0$ has 5 zeros inside $|z|<2$

c) Define $f(z) = 8z-1$; $h(z) = 2z^5$

For $|z|=1$ we have

$\boxed{3}$ $$|h(z)| = 2|z|^5 = 2$$

$$|f(z)| = |8z-1| \geq |8z|-1 = 7$$

Hence $|h(z)| < |g(z)|$ for $|z|=1$.

Hence: $2z^5+8z-1$ has 1 root inside $|z|<1$, because of the fact that $8z-1=0$ has a zero at $z=\frac{1}{8}$, with $|z|<1$.

$\boxed{3}$ This root of $2z^5+8z-1$ must be <u>real</u> and <u>positive</u> since at $z=0$ $2z^5+8z-1 = -1 < 0$

at $z=1$ $2z^5+8z-1 = 9 > 0$

So there must be a zero in the real interval $(0, 1)$.